Exercise 54

Find the absolute maximum and absolute minimum values of f on the given interval.

$$f(x) = \frac{x}{x^2 - x + 1}, \quad [0, 3]$$

Solution

Take the derivative of the function.

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left(\frac{x}{x^2 - x + 1}\right) \\
&= \frac{\left[\frac{d}{dx}(x)\right](x^2 - x + 1) - \left[\frac{d}{dx}(x^2 - x + 1)\right]x}{(x^2 - x + 1)^2} \\
&= \frac{(1)(x^2 - x + 1) - (2x - 1)x}{(x^2 - x + 1)^2} \\
&= \frac{1 - x^2}{(x^2 - x + 1)^2}
\end{aligned}$$

Set what's in the numerator equal to zero, and set what's in the denominator equal to zero. Solve both equations for x.

$$\begin{aligned}
1 - x^2 &= 0 & (x^2 - x + 1)^2 &= 0 \\
x^2 &= 1 & x^2 - x + 1 &= 0 \\
x = -1 \quad \text{or} \quad x &= 1 & x &= \frac{1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} \\
x = -1 \quad \text{or} \quad x &= 1 & x &= \frac{1 \pm i\sqrt{3}}{2}
\end{aligned}$$

The real numbers, $x = -1$ and $x = 1$, are the critical numbers. Only $x = 1$ is within $[0, 3]$, so evaluate f here.

$$f(1) = \frac{1}{1^2 - 1 + 1} = 1 \qquad \text{(absolute maximum)}$$

Now evaluate the function at the endpoints of the interval.

$$f(0) = \frac{0}{0^2 - 0 + 1} = 0 \qquad \text{(absolute minimum)}$$

$$f(3) = \frac{3}{3^2 - 3 + 1} = \frac{3}{7}$$

The smallest and largest of these numbers are the absolute minimum and maximum, respectively, over the interval $[0, 3]$.

The graph of the function below illustrates these results.